PICARD VARIETY OF AN ISOLATED SINGULAR POINT*

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1. *Introduction*

This paper concerns an attempt to generalize certain successful investigations on vector bundles on a projective manifold. Suppose that M is a projective manifold; let $H \xrightarrow{\pi} M$ be the (negative) hyperplane section bundle over M. Then the zero section of H can be reduced to a point, obtaining an analytic space $\tilde{H}$ with an isolated singularity $\{0\}$ and a holomorphic map $\tau\colon H \to \tilde{H}$. This can be arrived at in another way: if $M \subset \mathbf{P}^n$ and $\tau\colon \mathbf{C}^{n+1} - \{0\} \to \mathbf{P}^n$ is the natural projection, $\tilde{H} = \tau^{-1}(M) \cup \{0\}$. If $\xi \to M$ is a locally free sheaf, then $\tau_*\pi^*\xi$ is a coherent sheaf on $\tilde{H}$ which is locally free off zero. All information concerning locally free sheaves on M can be translated to information on such sheaves on $\tilde{H}$, and conversely. We are concerned with the study of such sheaves near a general isolated singular point; because of the above example and the close cohomological properties of coherent sheaves there is reason for optimism. In Section 2 we study those coherent sheaves near an isolated singular point which are invertible outside the singularity. The technique is Hironaka's resolution; we relate this group with the Picard variety of the resolved manifold, and that of the proper transform of the singular point and its normal bundle. This is possible because resolution transforms these sheaves (modulo torsion) into invertible sheaves. For coherent sheaves of rank greater than one no such assertion is possible. In Section 3 we discuss the possibility of transforming all coherent sheaves (modulo torsion) into locally free ones by proper modification. It is proven that for a fixed sheaf, such a proper modification exists.

Before proceeding we shall make our notions precise. An *isolated singularity* is a triple $\mathscr{X} = (X, \{x\}, \mathcal{O})$ where x is a point, X is a germ of a set and $\mathcal{O}$ is a ring for which there exists an analytic space $(\bar{X}, \bar{\mathcal{O}})$ for which $x \in \bar{X}$ and x is an isolated point of X_{sing}, X is the germ of $\bar{X}$ at x, and $\mathcal{O} = \bar{\mathcal{O}}_x$. A morphism $f\colon \mathscr{X} \to \mathscr{Y}$ is the germ at x of a holomorphic map $\bar{f}\colon (\bar{X}, \bar{\mathcal{O}}_X) \to (\bar{Y}, \bar{\mathcal{O}}_Y)$ such that $\bar{f}(x) = y$. We shall denote by IS the category of isolated singularities.

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If $\mathscr{X} = (X, \{x\}, \mathcal{O})$ is an isolated singularity we denote by $\mathscr{S}(\mathscr{X})$ the collection of germs of coherent $\mathcal{O}$-sheaves on X which are locally free outside of x. Two such sheaves are considered to be the same if they are isomorphic on $X - \{x\}$. $\mathscr{S}^r(\mathscr{X})$ is the collection of sheaves in $\mathscr{S}(\mathscr{X})$ which are of rank r on $X - \{x\}$. $\mathscr{S}'(\mathscr{X})$ is the collection of germs of locally free sheaves on $X - \{x\}$. Notice, that by definition, $\mathscr{S}^r(\mathscr{X}) \subset \mathscr{S}'^r(\mathscr{X})$. The Picard variety $\mathrm{Pic}(\mathscr{X})$ of $\mathscr{X}$ is $\mathscr{S}^1(\mathscr{X})$; we denote $\mathscr{S}'^1(\mathscr{X})$ by $\mathrm{Pic}'(\mathscr{X})$. Notice that if X is the germ of $(\bar{X}, \bar{\mathcal{O}})$ at x, then

$$\mathrm{Pic}'(\mathscr{X}) = \varinjlim_{\substack{U \to x \\ U \subset \bar{X}}} H^1(U - \{x\}, \bar{\mathcal{O}}^*).$$

2. *Study of* Pic *by Resolution*

Let $\mathscr{X} = (X, x, \mathcal{O})$ be an isolated singular point. By the resolution theorem of Hironaka [*4*] we can choose the representative X judiciously so that we can find a strongly pseudoconvex manifold $\tilde{X}$ together with a proper map $\pi \colon \tilde{X} \to X$ such that

(i) $\pi \colon \tilde{X} - \pi^{-1}(\{x\}) \cong X - \{x\}$, and

(ii) $\pi^{-1}(\{x\})$ is a finite union of projective manifolds of codimension one with only normal crossings which is negatively embedded (i.e., the zero section of the normal bundle of the embedding $\pi^{-1}(\{x\}) \to \tilde{X}$ is exceptional). We would like to show that $\mathscr{S}(\mathscr{X})$ is determined by the locally free sheaves on a suitable choice of $\tilde{X}$, and better yet just by these sheaves on a finite neighborhood of $\pi^{-1}(\{x\})$. This can be done for $\mathrm{Pic}(\mathscr{X})$ just because it is a cohomology group. We need the following theorem, a proof of which can be found in [*11*].

2.1. Theorem. *Let Z be a strongly pseudoconvex manifold with exceptional set E. Let $\mathscr{F} \to Z$ be a coherent analytic sheaf and p a positive integer. There is an ideal sheaf $\mathscr{I}$ such that $\{x;\, \mathscr{I}_x \neq \mathcal{O}_x\} = E$ and the natural restriction map $H^p(Z, \mathscr{F}) \to H^p(E, \mathscr{F} \otimes_{\mathcal{O}} \mathcal{O}/\mathscr{I})$ is injective.*

2.2. Corollary. *Let U, V be strongly pseudoconvex manifolds such that $V \subset U$ and for E the exceptional set of U, $E \subset V$. Let $\mathscr{F} \to U$ be a coherent sheaf. Then the restriction map $H^p(U, \mathscr{F}) \to H^p(V, \mathscr{F})$ is an isomorphism, $p > 0$.*

Proof. Because of the theorem the map is trivially injective. Let now $\omega \in H^p(V, \mathscr{F})$. Let K be a compact set in V, $\operatorname{int} K \supset E$. We may find a cochain $\theta \in C^p(U, \mathscr{F})$ such that $\theta \mid K = \omega \mid K$. Then $\delta\theta = 0$ near E, so by the theorem there is $\phi \in C^p(U, \mathscr{F})$, $\delta\theta = \delta\phi$. Then $\theta - \phi$ defines a class in $H^p(U, \mathscr{F})$, and $\theta - \phi \mid V = \omega + \alpha$, where $\delta\alpha = 0$ and $\alpha = 0$ near E. Thus α is a coboundary, so $\theta - \phi \mid V = \omega$ in $H^p(V, \mathscr{F})$.

2.3. Theorem. *Let $\mathscr{X} \in IS$. Let $\tilde{X}$ be a resolution of singularities of a representative $(X, x, \mathcal{O})$ of $\mathscr{X}$. There are arbitrarily small neighborhoods U of x such that the direct image map*

$$H^1(U, \mathcal{O}^*) \to \mathrm{Pic}(\mathscr{X})$$

is surjective. The kernel of this map is {locally free sheaves $\mathscr{S} \to U$ such that $\mathscr{S} \big| (U - \pi^{-1}(\{x\}))$ is free}.

Proof. The description of the kernel is obvious; a fuller description is given in [7]. There are arbitrarily small strongly pseudoconvex neighborhoods of x which are contractible to $\{x\}$. Let $U_0 \supset V_0$ be two such neighborhoods and $U = \pi^{-1}(U_0)$, $V = \pi^{-1}(V_0)$. Then all the restriction maps $H^p(U, Z) \to H^p(V, Z)$, $H^p(U, \mathcal{O}) \to H^p(V, \mathcal{O})$ $(p > 0)$ are isomorphisms; thus also $H^1(U, \mathcal{O}^*) \to H^1(V, \mathcal{O}^*)$ is an isomorphism. Letting now $V_0 \to \{x\}$,

$$H^1(U, \mathcal{O}^*) = \varinjlim_{\pi^{-1}(K) \subset V} H^1(V, \mathcal{O}^*).$$

Now the latter maps onto $\mathrm{Pic}(\mathscr{X})$. Let V_0 be some neighborhood of $\{x\}$ on which a given sheaf $\mathscr{S} \in \mathrm{Pic}(\mathscr{X})$ is defined. By the coherence at x, we may assume $\mathscr{S}$ has a non-zero section σ on V_0. Let $\mathfrak{U} = \{U\}$ be a coordinate cover for $\mathscr{S} \big| V_0 - \{x\}$, with transition functions $f_{UV} \in H^0(U \cap V, \mathcal{O}^*)$, and let σ be given by the cochain $\sigma_U \in H^0(U, \mathcal{O})$ with respect to this cover. The set $Z_0 = \bigcup_{U \in \mathfrak{U}} \{p \in U; \sigma_U(p) = 0\} \cup \{x\}$ is a subvariety of V_0 of codimension 1. Let $\pi^{-1}(V_0) = V$, $\pi^{-1}(\{x\}) = K$, and Z be the proper transform of Z_0 (the closure in V of $Z_0 - \{x\}$). $Z \cap K$ is of codimension 2 in V. For $x \in K$ we can find a suitable polydisk neighborhood U_x of x and irreducible functions $p_1, \cdots, p_S \in H^0(U_x, \mathcal{O})$ such that $Z = \bigcup \{p_i = 0\}$. Let $U \in \mathfrak{U}$. Then in $U \cap U_x$, $\{\sigma_U = 0\} = \bigcup \{p_i = 0\}$, so (when this set is non-empty) there are integers $k_1, \cdots, k_S$ such that $(\prod p_i^{k_i} \sigma_U^{-1})$ is an invertible holomorphic function in $U \cap U_x$ [3, p. 90]. It is easy to see, using the connectivity of the sets $\{p_i = 0\}$ that these integers are independent of U. Thus, letting $f_x = \prod p_i^{k_i}$ we see that $f_x f_y^{-1}$ is holomorphic in $U_x \cap U_y \to K$. Since it is also holomorphic at points $z \in K$ where $f_y(z) \neq 0$, it is holomorphic in $U_x \cap U_y - (K \cap Z)$ so by the extension theorem [3, p. 22], $f_{xy} = f_x f_y^{-1} \in H^0(U_x \cap U_y, \mathcal{O}^*)$. Letting $\mathscr{F}$ be the locally free sheaf given by these transition functions on the cover $\{U_x\}$, it is just a matter of tracing back to verify that $\mathscr{F} = \mathscr{S}$ on $V - K$, so $\pi_*(\mathscr{F}) = \mathscr{S}$.

This proof is an explicit form of the following more meaningful assertion: $\mathrm{Pic}(\mathscr{X})$ corresponds to the germs of divisors of $X - \{x\}$, and these lift to divisors of $\tilde{X}$ which is the germ at K of $H^1(\tilde{X}, \mathcal{O}^*)$.

Now we consider the group $H^1(X,\mathcal{O}^*)$ of line bundles on a strongly pseudoconvex manifold X which is sufficiently close to its exceptional set K. If X is contractible to K, the topological content of $H^1(X,\mathcal{O}^*)$ is easily related to that on K; thus attention focuses on the subgroup of topologically trivial line bundles, $\exp H^1(X,\mathcal{O})$.

The simplest possible case is that where K is a projective variety and X is the hyperplane section bundle over K. More generally, suppose that X is the space of a negative line bundle over K (here we equate K with the zero section of X). The computation of $H^1(X,\mathcal{O})$ has been done by Grauert in [2]. For any integer ν, X^ν denotes the $|\nu|$th tensor power of $X^{\operatorname{sign}\nu}$.

2.4. Theorem (Grauert). *Let $X \overset{\pi}{\to} K$ be a negative line bundle over the compact space K. There is an integer $\nu_0 \geqq 0$ such that*

$$H^1(X,\mathcal{O}) \cong \sum_{\nu=0}^{\nu_0} \oplus H^1(K,X^{-\nu}).$$

2.5. Corollary. *Under the same circumstances*

$$H^1(X,{}_X\mathcal{O}^*) \cong H^1(K,{}_X\mathcal{O}^*) \oplus \mathbf{C}^d,$$

where

$$d = \sum_{\nu>0} \dim H^1(K,X^{-\nu}).$$

Proof. Let $\xi \in H^1(X,\mathcal{O}^*)$. $\xi_0 \in H^1(K,\mathcal{O}^*)$ is the restriction of ξ to K. Then $\xi' = \xi\pi^*(\xi_0)^{-1} \in \exp H^1(X,\mathcal{O})$, since it has zero Chern class. In fact, since $\xi' \mid K = 1$, $\xi' = \exp\omega$, where $\omega \in H^1(X,\mathcal{I})$ ($\mathcal{I}$ the ideal sheaf of K). If we let Φ be the isomorphism of Grauert in Theorem 2.4, then the correspondence $\xi \to \xi_0 \oplus \Phi(\omega)$ is the desired map of 2.5. Because of Theorem 2.4 this map is isomorphic.

In the general situation we no longer have a holomorphic map $\pi\colon X \to K$ with which to pull back bundles, nor such a nice representation of $\mathcal{O}$ along K, so the above theorem seems to fail, although I know no counterexample. The following immediate corollary of Theorem 2.1 is the best assertion we can make:

2.6. Corollary. *Let X be a strongly pseudoconvex manifold with exceptional set K with ideal sheaf $\mathcal{I}$. There is a $\nu \geqq 0$ such that $H^1(X,\mathcal{O}) \to H^1(K,\mathcal{O}/\mathcal{I}^\nu)$ is injective. In particular,*

$$\dim H^1(X,\mathcal{O}) \leqq \sum_{j=0}^{\nu} \dim H^1(K,\mathcal{N}^j),$$

where $\mathcal{N}$ is the normal bundle of the embedding $K \subset X$.

Now we consider Pic′ $(\mathscr{X})$, i.e., the germ at $\{x\}$ of $H^1(X - \{x\}, \mathcal{O}^*)$. If $\dim X = 2$, there is not much to be said. For example, if $X = \mathbf{C}^2$, since $\dim H^1(\mathbf{C}^2 - \{0\}, \mathcal{O})$ is infinite, $\mathbf{C}^2 - \{0\}$ has many bad line bundles which cannot be extended "meromorphically" across zero. For example $\exp(1/zw)$ is the transition function for a bundle defined on the cover $\{z \neq 0\}$, $\{w \neq 0\}$. We shall show that in higher dimensional cases such extension is possible.

2.7. Theorem. *Let X be a strongly pseudoconvex manifold with exceptional set K, and $\dim X \geqq 3$. Let $\mathscr{M}^\nu$ be the sheaf of germs of meromorphic functions on X with poles of order $\leqq \nu$ only along K. Then the restriction map $H^1(X, \mathscr{M}^\nu) \to H^1(X - K, \mathcal{O})$ is surjective for ν large enough.*

Proof. Let $K - \{z_1 = 0, \cdots, z_S = 0\}$ where the z_i are distinct. By the finiteness theorems of Andreotti and Grauert [*1*], $\dim_{\mathbf{C}} H^1(X - K, \mathcal{O}) < \infty$. Since $H^1(X - K, \mathcal{O})$ is an $\mathcal{O}(X)$-module, there is an ideal I of finite codimension such that $I \cdot H^1(X - K, \mathcal{O}) = 0$. Since I is of finite codimension, $|I|$ consists of K together with finitely many points, and as in [*10*], we may exclude the other points. Thus $|I| = K$, so there is a $\nu \geqq 0$ such that $I \supset \{z_1^\nu, \cdots, z_S^\nu\}$.

Let $\omega \in H^1(X - K, \mathcal{O})$. Then (representing cohomology by any suitable chain complex), for each i there is $\phi_i \in C^0(X - K, \mathcal{O})$, $\delta\phi_i = z_i^\nu \omega$. Let $f_{ij} = (z_i z_j)^\nu (z_i^{-\nu}\phi_i - z_j^{-\nu}\phi_j)$. $\delta f_{ij} = 0$, so f_{ij} is holomorphic in $X - K$, and by Hartogs' theorem extends holomorphically to all of X. Consider now the covering by $U_i = \{z_i \neq 0\}$, $1 \leqq i \leqq s$. On U_i, $\omega = \delta z_i^{-\nu}\phi_i$, so ω is represented on the cover $\{U_i\}$ by $\{(z_i z_j)^{-\nu} f_{ij}\}_{1 \leqq i,j \leqq s}$. But for fixed i,j, this function is in $H^0(U_i \cap U_j, \mathscr{M}^\nu)$, so defines a class in $H^1(X, \mathscr{M}^\nu)$ which maps onto ω.

In the special case where K is a negatively embedded manifold in X, we can apply Kodaira's vanishing theorem [*6*] to K to obtain

2.8. Theorem. *Let K be a negatively embedded manifold in X. Suppose $H^2(X, \mathcal{O}) = 0$. Let $\mathscr{X}$ be the isolated singularity obtained by blowing down K. Then* $\mathrm{Pic}(\mathscr{X}) \cong H^1(X - K, \mathcal{O}^*)$.

Proof. We have a natural map of $\mathrm{Pic}(\mathscr{X}) \to H^1(X - K, \mathcal{O}^*)$ which by definition is an injection. We have to prove it is surjective, which by Theorem 2.3 amounts to proving that $H^1(X, \mathcal{O}^*) \to H^1(X - K, \mathcal{O}^*)$ is surjective. Since $H^2(X, \mathbf{Z}) \cong H^2(K, \mathbf{Z}) \cong H^2(X - K, \mathbf{Z})$ (the latter since $X - K$ is topologically a circle bundle over K with non-vanishing Chern class), we need only show that $H^1(X, \mathcal{O}) \to H^1(X - K, \mathcal{O})$ is surjective. Now the exactness of the sequence of sheaves

$$0 \to \mathscr{M}^i \to \mathscr{M}^{i+1} \to \mathscr{N}^{i+1} \to 0$$

where $\mathcal{N}$ is the normal bundle of the embedding $K \to X$ gives the exactness of

$$H^1(X, \mathcal{M}^i) \xrightarrow{\pi_i} H^1(X, \mathcal{M}^{i+1}) \longrightarrow H^1(X, \mathcal{N}^{i+1}).$$

By Kodaira's vanishing theorem, $H^1(X, \mathcal{N}^{i+1}) = 0$, so all the maps π_i are surjective, thus $H^1(X, \mathcal{O}) \to H^1(X, \mathcal{M}^\nu)$ is surjective. This together with Theorem 2.7 gives the surjectivity of $H^1(X, \mathcal{O}) \to H^1(X - K, \mathcal{O})$.

Remarks. The requirement that $H^2(X, \mathcal{O}) = 0$ is needed so that every topological line bundle contains an analytic representative and therefore the above argument works. Without this assumption, one is left with the interesting but unanswered question: how can we distinguish those topological line bundles on X which have an analytic representative?

Furthermore, since our interest is in $\mathrm{Pic}(\mathscr{X})$ for a given $\mathscr{X}$ we should find a good resolution X with exceptional set E so that the normal bundle of $E \subsetneq X$ is negative. Does Kodaira's vanishing theorem still hold, even though E is not a manifold?

3. *Proper Transformation of Coherent Sheaves*

The constructions of this section are so close to the concept of monoidal transformation of ideal sheaves as discussed in [8] and [5], that I suspect what I have to say that is not conjectural is already well known. Such readers as may be inclined to this view may look upon this as an attempted lesson in communication.

3.1. Proposition. *Let $\mathscr{S}$ be a coherent analytic sheaf on the analytic space X. The set $\sigma(\mathscr{S}) = \{x \in X; \mathscr{S}_x$ is not a free $\mathcal{O}_x$-module$\}$ is a proper subvariety of X.*

Proof. Let $x \in X$. There is by coherence a neighborhood U of x and an exact sequence $\mathcal{O}^p \xrightarrow{h} \mathcal{O}^q \xrightarrow{\pi} \mathscr{S} \to 0$ defined on U. Let $r = \max\{\mathrm{rank}_x h(x)\}$ (considering h as a holomorphic map into $L(\mathbf{C}^p, \mathbf{C}^q)$). We shall show that in U, $\sigma(\mathscr{S}) = \{x \in U; \mathrm{rank}\, h(x) < r\}$. This latter set is the zero locus of all determinants of $r \times r$ submatrices of h, and thus is a variety.

Now, if $\mathrm{rank}\, h(x) = r$, this is true in a neighborhood of x, so $\mathscr{S}$ is the cokernel (near x) of a regular homomorphism, and hence is free. On the other hand if $\mathscr{S}$ is free near x, then the given exact sequence becomes

$$\mathcal{O}^p \xrightarrow{\ h\ } \mathcal{O}^q \xrightarrow{\ \pi'\ } \mathcal{O}^s \longrightarrow 0$$

in a neighborhood V of x. In V, $\mathrm{rank}\, \pi' = s$, so $\ker \pi' = \mathrm{im}\, h$ is locally free in V. Then, for all $y \in V$, $\operatorname{rank} h(y) = \dim \operatorname{im} h(y)$ is independent of y, so is r, and thus $\operatorname{rank} h(x) = r$.

3.2. Definition. For $\mathscr{S}$ a coherent sheaf on an irreducible space X, define $\operatorname{rank} \mathscr{S}$ as the rank of the locally free sheaf $\mathscr{S} \big| X - \sigma(\mathscr{S})$.

Notice that the torsion sheaves are precisely the sheaves of rank 0. Our investigation of coherent sheaves will be done (as above for the Picard varieties) modulo sheaves of rank 0. We will show that given $\mathscr{S}$ we can find a proper transform $\tilde{X}$ of X so that the lift of $\mathscr{S}$ to $\tilde{X}$ is locally free (modulo torsion). The following definition is Remmert's [9].

3.3. Definition. Let X, Y be analytic spaces. A *meromorphic map* $F: X \to Y$ is a holomorphic map $F: X - V \to Y$ where V is a proper subvariety of X such that the closure in $X \times Y$ of the graph of F is a variety.

We shall speak of meromorphic maps into $G(d,n)$, the manifold of d-dimensional subspaces of $\mathbf{C}^n$. There is a canonical sequence of vector bundles over $G(d,n)$:

$$0 \longrightarrow \xi_d \longrightarrow \mathbf{C}^n \longrightarrow \eta^{n-d} \longrightarrow 0,$$

where, for $v \in G(d,n)$, $\xi_{d,v}$ is precisely the space v.

We shall refer to the following coordinatization of $G(d,n)$. Let π be a projection operator on $\mathbf{C}^n$ so that $K = \ker \pi$ has dimension d. Let $G = \operatorname{im} \pi$. Let $U_\pi = \{V; V \text{ is a complement to } G\}$. Then $G(d,n) = \bigcup_\pi U_\pi$, and U_π is a coordinate neighborhood with coordinates in $L(K,G)$, the linear maps of K into G. The correspondence $L(K,G) \to U_\pi$ is given by

$$f \to \{x + f(x); x \in K\}.$$

If $V \in U_\pi$, then $(1 - \pi)$: $V \to K$ is an isomorphism and we can take $f = \pi \cdot ((1 - \pi) \big| V)^{-1}$.

3.4. Proposition. *Let X be an analytic space, $\mathscr{S}$ a sheaf on X of rank r. Suppose we have a surjective map ϕ: $\mathcal{O}^p \to \mathscr{S}$. Corresponding to ϕ is a meromorphic map $F: X \to G(p - r, r)$, so that $F^*(\mathcal{O}^p \to \eta^r \to 0)$ is the given sequence $\mathcal{O}^p \to \mathscr{S} \to 0$ over $X - \sigma(\mathscr{S})$.*

Proof. Over $X - \sigma(\mathscr{S})$, $\mathscr{S}$ is the sheaf of sections of a vector bundle S and the given map arises from a regular bundle map

$$\mathbf{C}^p \xrightarrow{\ h\ } S \longrightarrow 0.$$

For $x \notin \sigma(\mathscr{S})$, $K_x = \ker h(x)$ is a $p - r$ dimensional subspace of $\mathbf{C}^p$. Define $F: X - \sigma(\mathscr{S}) \to G(p - r, p)$, $F(x) = K_x$. Clearly F has the desired property; we need only show it is meromorphic.

Fix $x_0 \in X$; by coherence we can find a map $g\colon \mathcal{O}^q \to \mathcal{O}^p$ so that

$$\mathcal{O}^q \xrightarrow{g} \mathcal{O}^p \xrightarrow{\phi} \mathcal{S} \longrightarrow 0$$

is exact in a neighborhood U of x_0. We consider g as a holomorphic map of U into $L(\mathbf{C}^q, \mathbf{C}^p)$. Fix a projection π, and consider

$$V_\pi = \{(x,f) \in U \times U_\pi;\ \pi \cdot g(x) = f \cdot (1-\pi) \cdot g(x)\}.$$

Clearly V_π is a subvariety of $U \times U_\pi$. We need only show that V_π is, over $U - \sigma(\mathcal{S})$, precisely the part of the graph of F in $[U - \sigma(\mathcal{S})] \times U_\pi$. Suppose $(x, F(x)) \in [U - \sigma(\mathcal{S})] \times U_\pi$. Now $F(x) = \operatorname{Im} g(x) \in U_\pi$, so $1-\pi\colon \operatorname{Im} g \to K$ is an isomorphism. Let α be its inverse. Then

$$\begin{aligned} \operatorname{Im} g(x) &= \operatorname{Im}((1-\pi)\cdot g(x) + \pi \cdot g(x)) \\ &= \{y + \pi \cdot \alpha(y);\ y \in K\}; \end{aligned}$$

thus $(x, F(x))$ has the coordinate $(x, \pi \cdot \alpha)$ which is clearly in V_π.

Now suppose $(x,f) \in V_\pi$, $x \notin \sigma(\mathcal{S})$. Then, for $\omega \in \mathbf{C}^q$,

$$g(x)(\omega) = (1-\pi)g(x)(\omega) + \pi g(x)(\omega) = y + f(y), \text{ for } y = (1-\pi)g(x)(\omega),$$

so $\operatorname{Im} g(x) \subset K + f(K)$. Since $x \notin \sigma(S)$, $\operatorname{rank} g(x) = \dim \operatorname{Im} g(x)$, so we have equality, and the proposition is proved.

I think this proposition is of interest in itself, for it allows the possibility of parametrizing certain equivalence classes of ample sheaves by means of meromorphic maps into Grassmannians. The latter may be amenable to the techniques of Douady's thesis. For the purpose of studying $\mathcal{S}(\mathcal{X})$ as we studied $\operatorname{Pic}(\mathcal{X})$, this proposition leads to this result.

3.5. Theorem. *Let X be an irreducible analytic space, and $\mathcal{S}$ a coherent sheaf on X. There are: an irreducible space $\tilde{X}$, proper map $\pi\colon \tilde{X} \to X$, locally free sheaf $\tilde{\mathcal{S}} \to \tilde{X}$, and $\mathcal{O}_{\tilde{X}}$-homomorphism $\phi\colon \pi^*(\mathcal{S}) \to \tilde{\mathcal{S}}$ such that*

(i) $\pi \,|\, \tilde{X} - \pi^{-1}(\sigma(\mathcal{S}))$ *is biholomorphic,*

(ii) ϕ *is surjective and* $\ker \phi$ *is of rank* 0.

Proof. First we shall construct the required objects locally and then show that they are independent of the choices made. Then the global objects can be constructed by patching.

So we may suppose that there is a surjective sheaf map $\mathcal{O}^p \xrightarrow{\beta} \mathcal{S} \to 0$. Let $r = \operatorname{rank} \mathcal{S}$. Let $F\colon X \to G(p-r, r)$ be the corresponding meromorphic map given by Proposition 3.4, and let $\tilde{X}$ be the closure of the graph of F in $X \times G$. Let $\pi\colon X \times G \to X$, $\tau\colon X \times G \to G$ be the natural projections. The π above is $\pi \,|\, \tilde{X}$, and $\tilde{\mathcal{S}} = \tau^*(\eta) \,|\, \tilde{X}$.

The requirement (i) is obvious. On $\tilde{X}$ we have the canonical exact sequence (lifted by τ)

$$\mathcal{O}^p \xrightarrow{\alpha} \tilde{\mathcal{S}} \longrightarrow 0,$$

and the exact sequence $\mathcal{O}^p \xrightarrow{\beta^*} \pi^*(\mathcal{S}) \to 0$. By Proposition 3.4 these are the same on $\tilde{X} - \pi^{-1}(\sigma(\mathcal{S}))$. Thus, since $\tilde{\mathcal{S}}$ is locally free α annihilates $\ker \beta$, so $\phi = \alpha \cdot (\beta^*)^{-1} : \pi^*(\mathcal{S}) \to \tilde{\mathcal{S}}$ is well-defined and obviously surjective. The kernel of ϕ is clearly the torsion subsheaf of $\pi^*(\mathcal{S})$ (for ϕ is isomorphic off $\pi^{-1}(\sigma(\mathcal{S}))$), so (ii) is proven.

Now we verify that these are defined independently of the choice of generators of $\mathcal{S}$. Suppose $\mathcal{O}^s \xrightarrow{\beta'} \mathcal{S} \to 0$ is another exact sequence on X. If we shrink X we may assume that there is a map $\gamma : \mathcal{O}^s \to \mathcal{O}^p$ such that

$$\begin{tikzcd} \mathcal{O}^s \arrow[r, "\beta'"] \arrow[d, "\gamma"'] & \mathcal{S} \arrow[r] & 0 \\ \mathcal{O}^p \arrow[ur, "\beta"'] & & \end{tikzcd}$$

is commutative. We consider γ as an $L(\mathbf{C}^s, \mathbf{C}^p)$-valued holomorphic map on X. Consider the map

$$\theta : X \times \mathbf{C}^{s+p} \longrightarrow X \times \mathbf{C}^{s+p}$$

$$(x, (u, v)) \longrightarrow (x, (u, v - \gamma u)).$$

θ is non-singular and linear in the second variable and therefore defines a biholomorphic map $\theta : X \times G(s + p - r, s + p) \to X \times G(s + p - r, s + p)$. The map $G(p - r, p) \to G(s + p - r, s + p)$ given by $V \to \mathbf{C}^s \times V$ is an embedding; let $\bar{\theta} : X \times G(p - r, p) \to X \times G(s + p - r, s + p)$ be the embedding obtained by the composition of these two maps. (We should also notice that

$$\bar{\theta}^*(0 \to \xi_{p+s-r} \to \mathbf{C}^{p+s} \to \eta^r \to 0) = (0 \to \xi_{p-r} \oplus \mathbf{C}^s \to \mathbf{C}^{p+s} \to \eta^r \to 0)).$$

Now we have an exact sequence $\mathcal{O}^{s+p} \xrightarrow{\beta''} \mathcal{S} \to 0$ ($\beta'' = \beta' \oplus \beta$) to which we can apply Proposition 3.4 to obtain a meromorphic map $F'' : X \to G(s + p - r,\ s + p)$, and a corresponding transform $X'' \subset X \times G(s + p - r,\ s + p)$. We want to prove that $\bar{\theta} : \tilde{X} \cong X''$. That will suffice to prove that our solution obtained via β is isomorphic to that obtained by β''. We can apply the same arguments to the pair β' and β'', thus obtaining the desired isomorphism between the constructs of β and β'.

To prove that $\bar{\theta}: \tilde{X} \cong X''$ we need only show that if $(x, v) \in \Gamma_F$, then $\theta(x, v) \in \Gamma_{F''}$ (since $\bar{\theta}$ is an embedding). $\bar{\theta}$ is derived from the vector bundle map θ corresponding to the sheaf map (on X) $\theta: \mathcal{O}^{s+p} \to \mathcal{O}^{s+p}$ given by $\theta(u, v) = (u, v - \gamma u)$. We have to show that $\theta: \mathcal{O}^s \oplus \ker \beta \cong \ker \beta''$.

θ is clearly one-one. If $(u, v) \in \mathcal{O}^{s+p}$ with $\beta v = 0$, then

$$\beta''(\theta(u, v)) = \beta' u + \beta(-\gamma u + v) = \beta' u - \beta\gamma u + \beta v = 0.$$

θ is surjective. Suppose $\beta''(u, v) = \beta' u + \beta v = 0$. But $\beta' u = \beta(\gamma u)$, so $\beta(\gamma u + v) = 0$. Finally $(u, v) = \theta\,(u, \gamma u + v)$.

Remarks. Notice that $\tilde{\mathscr{S}}$ is nothing other than $\pi^*(\mathscr{S}/\mathscr{T})$, where $\mathscr{T}$ is the torsion sheaf of $\mathscr{S}$. The avowed purpose of this theorem is to study coherent sheaves modulo torsion by reducing this to the study of vector bundles on manifolds via resolution. I have not gone very far in such a program. One question which arises is this: suppose X is compact (or is a germ at a point); can we find one proper modification $\tilde{X}$ so that *every* coherent sheaf of rank r lifts to a locally free sheaf on $\tilde{X}$ (modulo torsion)? Another question: to what extent do the Grassmann varieties $\pi^{-1}(x)$; $x \in \sigma(\mathscr{S})$ determine the sheaf $\mathscr{S}$? Will a finite neighborhood suffice? Finally, is the construction of Theorem 3.5 determined by some universal property? If so, what is it?

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